Objectives:

- Use the Extreme Value Theorem to find a function's absolute extrema on a closed interval.
- Use the Second Derivative Test to categorize critical points.
- Practice finding absolute and local extrema.

From yesterday's project:

The Extreme Value Theorem

If f is **continuous** on a closed interval $[a, b]$, then f must attain an absolute maximum value, $f(c)$ and an absolute minimum value $f(d)$ for some numbers c and d in $[a, b]$.

NOTE: This does not mean c and d are unique. For example: constant functions, $\sin(x)/\cos(x)$.

Examples: Let's see how we can find absolute maximum and minimum values, if they exist, for the following functions over the given intervals.

1. $f(x) = x - \ln(x)$ on $[0, 2]$

   f is NOT continuous on this interval, since $\ln(x)$ is undefined at 0. So, we can't know for sure if f has absolute extrema. Let's look instead at an interval where f is continuous: $[0.1, 2]$.

   $f'(x) = 1 - \dfrac{1}{x}$ is defined everywhere in $[.1, 2]$, so only critical point is where $\dfrac{1}{x} = 1$, i.e. $x = 1$.
   Now we compare values for critical points and endpoints.
   So, absolute max ≈ 2.4 at $x = 0.1$ and absolute min $= 1$ at $x = 1$.

x	$f(x)$
0.1	$0.1 - \ln(0.1) \approx 2.4$
1	$1 - \ln(1) = 1$
2	$2 - \ln(2) \approx 1.3$

2. $g(t) = t^3 - 3t^2 - 20$ on $[3, 6]$

   $g'(t) = 3t^2 - 6t$. $g'(t)$ is defined everywhere on $[3, 6]$. $g'(t) = 3t(t - 2)$ so $g'(t) = 0$ when $t = 0, 2$. Neither of these critical points are in the interval we're concerned about. So, just test endpoints:

t	$g(t)$
3	$3(3^2) - 6(3) = 9$
6	$6(6^2) - 6(6) = 180$

   So, absolute max = 180 at t=6, absolute min = 9 at t=3.

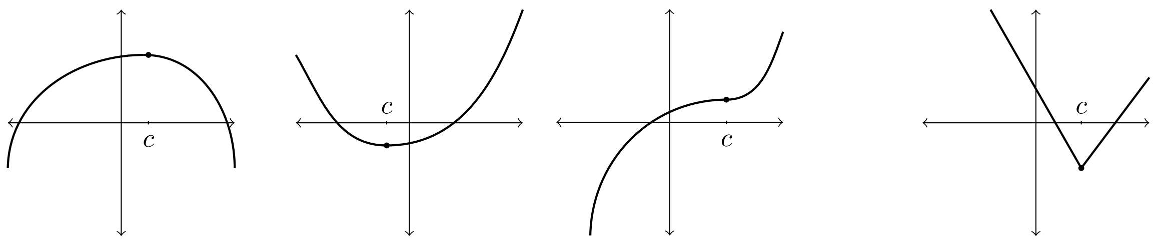
3. $h(x) = 3 - |x - 1|$ on $[0, 5]$

   $h'(x)$ is zero nowhere, undefined at $x = 1$. $x = 1$ is in the domain of $h(x)$, so it's a critical pt.

x	$h(x)$
0	$3 - \|0 - 1\| = 2$
1	$3 - \|1 - 1\| = 3$
5	$3 - \|5 - 1\| = -1$

   So absolute min = -1 at x=5, absolute max=3 at x=1.

More on Local Extrema:
What about functions not over a closed interval? Then we can't compare function values of critical points to function values at endpoints. How can we figure out if a critical point is a maximum, a minimum, or neither?
So far we have one tool to classify critical points, the first derivative test.
Now, we introduce another option:



$f(c)$ is a local max	$f(c)$ is local min	$f(c)$ is not an extremum	$f(c)$ is a a local min
$f'(c)$ =0	$f'(c)$ =0	$f'(c)$ =0	$f'(c)$ is undefined
$f''(c) < 0$	$f''(c) > 0$	$f''(c) = 0$	$f''(c)$ is undefined

The Second Derivative Test:
If f is continuous near c, then:

(a) If $f'(c) = 0$ and $f''(c) > 0$ then f has a local minimum at c.

(b) If $f'(c) = 0$ and $f''(c) < 0$ then f has a local maximum at c.

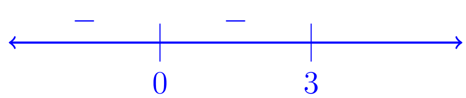
(c) If $f'(c) = 0$ and $f''(c) = 0$ no conclusion, use first derivative test instead.

(d) If $f'(c)$ or $f''(c)$ is undefined no conclusion, use first derivative test instead.

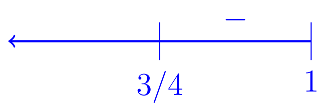
Examples: Find and classify all critical points of the following functions:

1. $f(x) = x^4 - 4x^3$

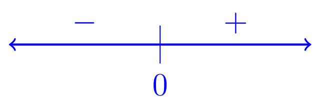
   The domain is $\mathbb{R} = (-\infty, \infty)$ so no endpoints. $f'(x) = 4x^3 - 12x^2 = 4(x^3 - 3x^2) = 4(x^2)(x-3)$ is defined everywhere so critical points are $x = 0, 3$.
   $f''(x) = 12x^2 - 24x$ so $f''(0) = 0$ and $f''(3) = 36$. Then we can conclude $f(3)$ is a local minimum.
   The 2nd derivative test is inconclusive for $x = 0$, so use first derivative test:

   $f'(x) = 4x^3 - 12x^2$ 

   $f(0)$ is not a local max or min.

2. $p(x) = x + \sqrt{1-x}$
   Domain is $x \leq 1$, so $x = 1$ is an endpoint. $p'(x) = 1 + \frac{1}{2}(1-x)^{-1/2}(-1)$ so critical points are $x = 1$ where $p'(x)$ is undefined and $x = \frac{3}{4}$ where $p'(x) = 0$.
   $p''(x) = -\frac{1}{4}(1-x)^{-3/2}$ so $p''\left(\frac{3}{4}\right) = \dfrac{-1}{4(1/2)^3} = -2$. Thus, $p\left(\frac{3}{4}\right)$ is a local max.
   For $x = 1$, p' is undefined, so can't use 2nd derivative test.
   $p'(x) = 1 - \dfrac{1}{2\sqrt{1-x}}$
   

   $p(x)$ is decreasing up to its endpoint $x = 1$, so $p(1)$ must be a local min.

3. $g(t) = t^4$
   $g'(t) = 4t^3$ so the only critical point is at $t = 0$. $g''(t) = 12t^2$ so $g''(0) = 0$. We might think from part 1 that this means there is no max/min but really it means we need another test - the first derivative test.
   $g'(t) = 4t^3$
   

   Since $g(t)$ switches from negative to positive at $t = 0$, $g(0)$ is a local min.

Finding Inflection Points:

An inflection point is a point on a curve where <u>the concavity of the curve changes</u>. We can also think of an inflection point as <u>a local maximum or minimum of the first derivative</u>.

To find an inflection point:

1. Find where the second derivative is <u>0 or undefined</u>.
2. Find the sign of the <u>second derivative</u> on each interval between the points from step 1.
3. If <u>the sign of the 2nd derivative changes at c</u>, then $(c, f(c))$ is an inflection point.

Example: Find and classify all critical points of $f(x) = 3xe^{-2x}$, as well as finding its inflection points.

$f'(x) = 3xe^{-2x}(-2) + 3e^{-2x} = 3e^{-2x}\big(-2x+1\big)$ so, $f'(x)$ is always defined and only zero at $x = \frac{1}{2}$. So, critical point is $x = \frac{1}{2}$.

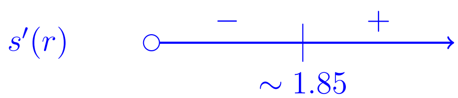
$f''(x) = 3e^{-2x}\big(-2\big) + 3e^{-2x}(-2)\big(-2x+1\big) = -6e^{-2x}\big(1-2x+1\big) = -6e^{-2x}\big(2-2x\big)$
So, $f''\left(\frac{1}{2}\right) = -6e^{-1}(2-1) = (-)(+)(+) = -$. Then we know $f\left(\frac{1}{2}\right)$ is a local maximum.

$f''(x) = 0$ when $-6e^{-2x}\big(2-2x\big) = 0$, so at $x = 1$. $f''(x)$ is never undefined. Then $(1, f(1)) = (1, 3e^{-2})$ is the only inflection point of f.

More Examples!

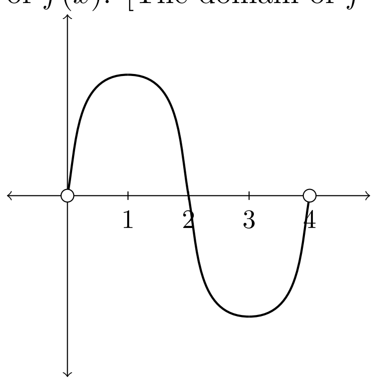
1. Consider $s(r) = 2\pi r^2 + \frac{80}{r}$ on the domain $(0, \infty)$. Find, if possible, the local and absolute extrema and inflection points of $s(r)$.

   $s'(r) = 4\pi r - \dfrac{80}{r^2}$ is defined everywhere on $(0, \infty)$, so only critical point is where $4\pi r = \dfrac{80}{r^2}$.
   Critical point: $r = \sqrt[3]{\dfrac{80}{4\pi}} = \sqrt[3]{\dfrac{20}{\pi}} \approx 1.85$. $s''(r) = 4\pi + \dfrac{160}{r^3}$ is always positive, so $s\left(\sqrt[3]{\dfrac{20}{\pi}}\right)$ is a local minimum. There are no local maximums.
   Since our interval is open, the endpoints can't be extrema (they're not in the domain). However, we can conclude that we do have an absolute minimum at $r = \sqrt[3]{\dfrac{20}{\pi}}$. In section 4.6 we will define precisely when we can make this conclusion, but consider the sign chart we can draw from our knowledge about the 1st and 2nd derivatives.

   If $s(r)$ decreases up to our local min and then increases forever, no value of $s(r)$ can be smaller than our local min.
   Since the second derivative is always positive, there is no inflection point.

2. Given the following graph of $f'(x)$, find the local extrema, absolute extrema, and inflection points of $f(x)$. [The domain of f is $(0, 4)$.] Use this information to graph $f(x)$.

   The only point where f' is zero or undefined on $(0, 4)$ is $x = 2$, so this is our one option for a max or min of f. On $(0, 2)$, $f'(x) > 0$ and on $(2, 4)$, $f'(x) < 0$ so by the first derivative test, $f(2)$ is a local maximum of f. Since $f'(x)$ has only one sign change, we can conclude that $f(2)$ is an absolute maximum on $(0, 4)$.
   The inflection points of $f(x)$ are where $f''(x)$ changes sign - a.k.a. the local maximums and minimums of $f'(x)$. So $f(x)$ has inflection points at $x = 1, 3$.